Recitation 3. March 16

Focus: nullspaces, systems of equations, dimension and rank, orthogonal subspaces, projection matrices

The **nullspace** of an $m \times n$ matrix A is the vector space consisting of those $\boldsymbol{v} \in \mathbb{R}^n$ such that $A\boldsymbol{v} = 0$. To find a **basis** of the nullspace, put the matrix in reduced row echelon form, and look at the **pivot and free variables**.

The general solution to a system of equations $A\boldsymbol{v} = \boldsymbol{b}$ is:

$$\boxed{\boldsymbol{v} = \boldsymbol{v}_{\text{particular}} + \boldsymbol{w}_{\text{general}}}$$

where $\boldsymbol{v}_{\text{particular}}$ is a particular solution, and $\boldsymbol{w}_{\text{general}}$ is a general element of $N(A)$.

The **dimension** of a vector space is the number of vectors in a basis (i.e. a collection of linearly independent vectors which span the vector space in question). The **rank** of a matrix A is the dimension of its column space $C(A)$.

Two subspaces V, W of $\mathbb{R}^n$ are called **orthogonal** if any vector in a basis of V is orthogonal (a.k.a. perpendicular, a.k.a. has dot product 0) to any vector in a basis of W. For any matrix A:

- the **column space** is the orthogonal complement of the **left nullspace**
- the **row space** is the orthogonal complement of the **nullspace**

The projection of a vector $\boldsymbol{b} \in \mathbb{R}^n$ onto a subspace $V \subset \mathbb{R}^n$ is the closest vector $\boldsymbol{p} \in V$ to $\boldsymbol{b}$. It can be computed by:

$$\boxed{\boldsymbol{p} = \underbrace{A(A^T A)^{-1} A^T}_{\text{projection matrix}} \boldsymbol{b}}$$

for any matrix A with column space V. The columns of A must be linearly independent to apply the formula above!

1. Use Gauss-Jordan elimination to compute the null space $N(X)$ of the matrix

$$X = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & -2 & 0 & 5 \\ -2 & 4 & -1 & -5 \end{bmatrix}$$

Then find the general solution to the system of equations:

$$X\boldsymbol{v} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \tag{1}$$

Solution: We perform Gauss-Jordan elimination on the augmented matrix of the system of equations (for the first part of the problem, we could have just done it for X itself, but there's no harm in doing it for the augmented matrix and it will save us some time later on):

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & -1 \\ 3 & -2 & 0 & 5 & 0 \\ -2 & 4 & -1 & -5 & -1 \end{array}\right] \xrightarrow{r_2 - 3r_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & -1 \\ 0 & -8 & 3 & 5 & 3 \\ -2 & 4 & -1 & -5 & -1 \end{array}\right] \xrightarrow{r_3 + 2r_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & -1 \\ 0 & -8 & 3 & 5 & 3 \\ 0 & 8 & -3 & -5 & -3 \end{array}\right]$$

$$\xrightarrow{r_3 + r_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & -1 \\ 0 & -8 & 3 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_2 \cdot (-\frac{1}{8})} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & -1 \\ 0 & 1 & -\frac{3}{8} & -\frac{5}{8} & -\frac{3}{8} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 - 2r_2} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{8} & -\frac{5}{8} & -\frac{3}{8} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: The nullspace consists of those vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that:

$$\begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{3}{8} & -\frac{5}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{or, in other words} \qquad \begin{cases} a = \frac{c-5d}{4} \\ b = \frac{3c+5d}{8} \end{cases}$$

The pivot variables are a and b, and the free variables are c and d. We conclude that the nullspace $N(X)$ consists of all vectors of the form:

$$\begin{bmatrix} \frac{c-5d}{4} \\ \frac{3c+5d}{8} \\ c \\ d \end{bmatrix} \tag{2}$$

Alternatively, a basis for the nullspace $N(X)$ is given by two vectors, each of which involves setting c (or d) equal to 1 and all other free variables equal to 0:

$$\text{a basis of } N(X) \text{ consists of the vectors } \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{5}{4} \\ \frac{5}{8} \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

Since the solution to systems of equations is unaffected by Gaussian or Gauss-Jordan elimination (but only if the right hand side of (1) is also involved in the elimination process), we might as well solve the system:

$$\begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{3}{8} & -\frac{5}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{3}{8} \\ 0 \end{bmatrix} \tag{4}$$

We know that the general solution is given by:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \boldsymbol{v}_{\text{particular}} + \boldsymbol{w}_{\text{general}}$$

where $\boldsymbol{w}_{\text{general}}$ is a general element of $N(X)$ (which we have already computed in (2)) and $\boldsymbol{v}_{\text{particular}}$ is a particular solution of (4). To construct such a particular solution, just expand the system (4):

$$\begin{cases} a - \frac{c}{4} + \frac{5d}{4} = -\frac{1}{4} \\ b - \frac{3c}{8} - \frac{5d}{8} = -\frac{3}{8} \end{cases}$$

To get a particular solution of the system above, just set the free variables c and d equal to arbitrary numbers (say $c = d = 0$) and compute:

$$a = -\frac{1}{4} \qquad \text{and} \qquad b = -\frac{3}{8}$$

We conclude that a particular solution is:

$$\boldsymbol{v}_{\text{particular}} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{3}{8} \\ 0 \\ 0 \end{bmatrix}$$

So the answer for the general solution is:

$$\begin{bmatrix} -\frac{1}{4} \\ -\frac{3}{8} \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -\frac{5}{4} \\ \frac{5}{8} \\ 0 \\ 1 \end{bmatrix}$$

2. Find bases for the four fundamental subspaces of the matrix X in Question 1, and check that these four subspaces are orthogonal complements of each other, in the appropriate pairs.

Solution: We have already computed a basis of the nullspace in (3), so its dimension is 2.

The column space is spanned by the pivot columns. Since the pivots are on the first and second columns, then:

$$\text{a basis of } C(X) \text{ consists of the vectors } \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \tag{5}$$

The dimension of the column space is therefore 2, and this by definition is also the rank of the matrix.

Since the row space is unaffected by row operations, we see that:

$$\text{a basis of } C(X^T) \text{ consists of the vectors } \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{4} \\ \frac{5}{4} \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ -\frac{3}{8} \\ -\frac{5}{8} \end{bmatrix} \tag{6}$$

The dimension of the row space is therefore 2.

Computing a basis of the left nullspace involves remembering the product of elimination and diagonal matrices that put X into reduced row echelon form (this product is what we called K at the end of Lecture 10):

$$\underbrace{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{3}{8} & -\frac{1}{8} & 0 \\ -1 & 1 & 1 \end{bmatrix}}_{\text{call this } K} X = \underbrace{\begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{3}{8} & -\frac{5}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{call this } R}$$

How did we get this value for K? Just keep track of all the row operations that we performed to get from X to its reduced row echelon form R, write those row operations as elementary matrices, and multiply them out:

$$K = E_{12}^{(-2)} D_2^{(-1/8)} E_{32}^{(1)} E_{31}^{(2)} E_{21}^{(-3)} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{3}{8} & -\frac{1}{8} & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

As we have seen at the end of Lecture 10, the left nullspace consists of $\boldsymbol{w}K$, where $\boldsymbol{w}$ goes over all row vectors which have all 0's in the pivot rows of X. Since the reduced row echelon form of X has pivots on rows 1 and 2, and the 3rd row is all zeroes, we conclude that the left nullspace consists of the vectors:

$$\begin{bmatrix} 0 & 0 & \lambda \end{bmatrix} K = \begin{bmatrix} -\lambda & \lambda & \lambda \end{bmatrix}$$

as λ goes over all the real numbers. Let's put these row vectors in column form (just because it's more standard to do so), and we conclude that:

$$\text{a basis of } N(X^T) \text{ consists of the vector } \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \tag{7}$$

It is easy to see that the subspaces $N(X)$ and $C(X^T)$ (respectively $C(X)$ and $N(X^T)$) are complementary because their dimensions sum up to $2 + 2 = 4$ (respectively $2 + 1 = 3$). To show that these subspaces are mutually orthogonal, just show that each of the basis vectors in (3) is orthogonal to each of the basis vectors in (6), and that each of the basis vectors in (5) is orthogonal to the basis vector in (7). This is a straightforward dot product computation, and we leave it as an exercise to you.

3. Consider the subspace V with basis given by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Compute the closest point of V to the vector $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

   Check that the projection matrix P_V onto the subspace V satisfies that $P_V^2 = P_V$.

Solution: We compute the projection matrix P_V. For this we use $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$. First compute:

$$A^T A = \begin{bmatrix} 14 & 4 \\ 4 & 2 \end{bmatrix}$$

Then we can compute, using the formula for the inverse of 2×2 matrices from PSet 2:

$$(A^T A)^{-1} = \frac{1}{12} \begin{bmatrix} 2 & -4 \\ -4 & 14 \end{bmatrix}$$

Then we have:

$$P_V = A(A^T A)^{-1} A^T = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

One can check $P_V^2 = P_V$ by hand. Then we can compute the closest point $\boldsymbol{p}$ to $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ in V by computing:

$$\boldsymbol{p} = \text{proj}_V \left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right) = P_V \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$